Question One: (15 Marks) *Start a new sheet of paper.*

a) Find $\int \frac{x}{\sqrt{2-x^2}}dx$ using the substitution $x=\sqrt{2}\sin\theta$. [2]

b) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$, and hence find

$\int \sin 5x \cos 3x dx$. [3]

c) Use Integration by Parts to show that $\int_0^1 \tan^{-1} x dx = \frac{\pi}{4}-\frac{1}{2}\ln 2$. [3]

d) Given that $J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$:

i) Prove that $J_n = \frac{(n-1)}{n}J_{n-2}$, where n is an integer and $n \geq 2$. [4]

ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x dx$. [3]

Question Two: (15 Marks) *Start a new sheet of paper.*

a) Given that $z=2+i$ and $\omega = 2-3i$, find, in the form $a+ib$

i) $(\overline{z})^2$ [1]

ii) $\left(\frac{z}{\omega}\right)$ [1]

b) On the Argand diagram, shade the region where the inequalities $-1<|z|<1$ and $\frac{\pi}{4}<\arg(z)<\frac{3\pi}{4}$ both hold. [3]

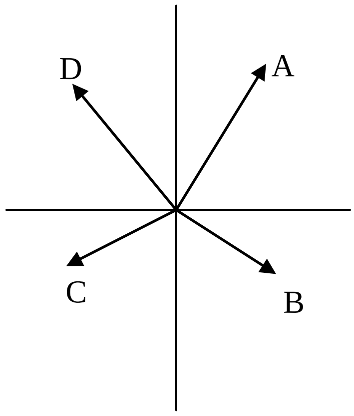
c) Find the complex square roots of $7+6i\sqrt{2}$, giving your answer in the form $a+ib$, where a and b are real. [3]

(Question 2 continued over)

d) Given the two complex numbers $z_1 = r_1 cis\theta$ and $z_2 = r_2 cis\phi$,

i) Show that, if z_1 and z_2 are parallel, $z_1 = kz_2$, for k real. [1]

ABCD is a quadrilateral with vertices A, B, C and D represented by the complex numbers (vectors) z_1, z_2, z_3 and z_4, as shown in the sketch opposite.



ii) Give two possible vectors (in terms of z_1, z_2) for side AB. [1]

iii) If ABCD is a parallelogram, show that $z_1 - z_2 - z_3 + z_4 = 0$. [3]

e) Explain the fallacy in the following argument: [2]

$-1 = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Hence $1 = -1$.

Question Three: (15 Marks) *Start a new sheet of paper.*

a) $F(x)$ is defined by the equation $f(x) = x^2\left(x - \frac{3}{2}\right)$, on the domain $-2 \le x \le 2$.

Note: each sketch below should take about one third of a page.

i) Draw a neat sketch of $F(x)$, labelling all intersections with coordinate axes and turning points. [2]

ii) Sketch $y = \dfrac{1}{F(x)}$ [2]

iii) Sketch $y = \sqrt{F(x)}$ [2]

iv) Sketch $y = \ln(F(|x|))$ [2]

b) The Hyperbola $\mathcal{H}$ has the equation $\frac{x^2}{25}-\frac{y^2}{9}=1$.

i) Find the eccentricity of $\mathcal{H}$. [1]

ii) Find the coordinates of the foci of $\mathcal{H}$. [1]

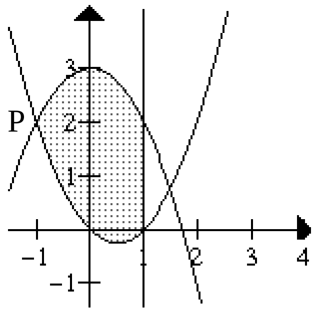
iii) Draw a neat one third of a page size sketch of $\mathcal{H}$. [2]

iv) The line $x=6$ cuts $\mathcal{H}$ at A and B. Find the coordinates of A and B if A is in the first quadrant. [1]

v) Derive the equation of the tangent to $\mathcal{H}$ at A. [2]

Question Four: (15 Marks) *Start a new sheet of paper.*

a) The shaded region bounded by $y=3-x^2$, $y=x^2-x$ and $x=1$ is rotated about the line $x=1$. The point P is the intersection of $y=3-x^2$ and $y=x^2-x$ in the second quadrant.



i) Find the x coordinate of P. [1]

ii) Use the method of cylindrical shells to express the volume of the resulting solid of resolution as an integral. [3]

iii) Evaluate the integral in part (ii) above. [2]

(Question 4 continued over)

b) Find real numbers A, B and C such that
$$\frac{x}{(x-1)^2(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-2)}.$$ [2]

Hence show that $\displaystyle\int_0^{\frac{1}{2}}\frac{x}{(x-1)^2(x-2)}dx=2\ln\left(\frac{3}{2}\right)-1$. [2]

c) Find all x such that $\cos 2x=\sin 3x$, if $0\le x\le\frac{\pi}{2}$. [2]

d) Solve for x: $\tan^{-1}(3x)-\tan^{-1}(2x)=\tan^{-1}\left(\frac{1}{5}\right)$ [3]

<u>Question Five:</u> (15 Marks) *Start a new sheet of paper.*

a) For the polynomial equation $x^3+4x^2+2x-3=0$ with roots α,β and γ, find:

i) The value of $\alpha^2+\beta^2+\gamma^2$ [1]

ii) The equation whose roots are $(1-\alpha),(1-\beta),(1-\gamma)$. [2]

iii) The equation whose roots are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$. [3]

b) Determine all the roots of $8x^4-25x^3+27x^2-11x+1=0$ given that it has a root of multiplicity 3. [4]

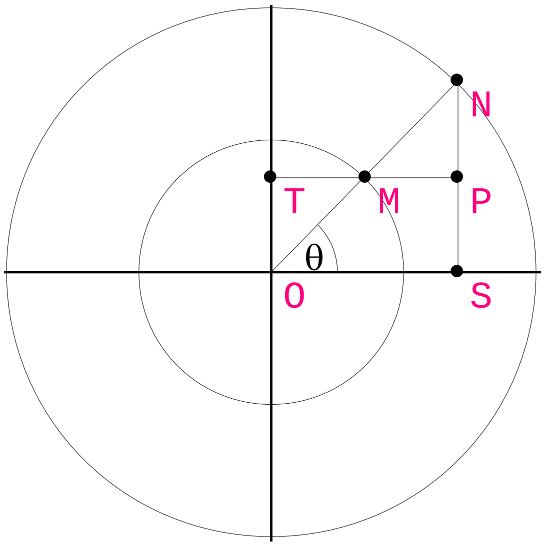
c) The equation $x^4+4x^3+5x^2+2x-20=0$ has roots α,β,γ and δ over the complex field.

i) Show that the equation whose roots are $\alpha+1,\beta+1,\gamma+1$ and $\delta+1$ is given by $x^4-x^2-20=0$. [2]

ii) Hence solve the equation $x^4+4x^3+5x^2+2x-20=0$. [3]

<u>Question Six:</u> (15 Marks) *Start a new sheet of paper.*

a)



The circles above have centres at O and radii of 5 units and 3 units respectively.

A ray from O making an angle θ with the positive x-axis, cuts the circles at the points M and N as shown.

NS is drawn parallel to the y-axis and MT parallel to the x-axis.

NS and MT intersect at P.

i) Show that the parametric equations of the locus of P in terms of θ are given by $x = 5\cos\theta$ and $y = 3\sin\theta$. [2]

ii) By eliminating θ, find the Cartesian equation of this locus. [1]

iii) Find the equation of the normal (in general form) at the point P when $\theta = \dfrac{\pi}{3}$. [2]

b) The functions $S(x)$ and $C(x)$ are defined by the formulae
$s(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ and $c(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$.

i) Verify that $S'(x) = C(x)$. [1]

ii) Show that $S(x)$ is an increasing function for all real x. [1]

iii) Prove $[C(x)]^2 = 1 + [S(x)]^2$ [2]

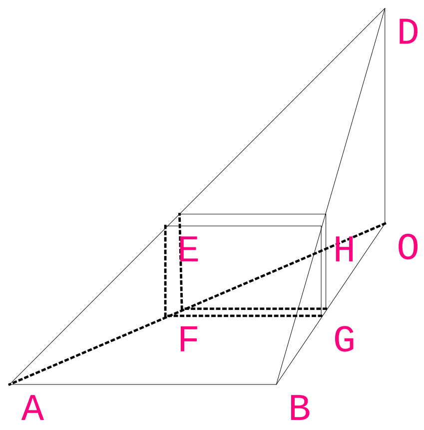
iv) $S(x)$ has an inverse function, $S^{-1}(x)$, for all real values of x. Briefly justify this statement. [1]

v) Let $y = S^{-1}(x)$. Prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$. [2]

vi) Hence, or otherwise, show that $S^{-1}(x) = ln\{x + \sqrt{1+x^2}\}$. [3]

<u>Question Seven:</u> (15 Marks) *Start a new sheet of paper.*

a) Let OAB be an isosceles triangle, OA = OB = r, AB = b.



Let OABD be a triangular pyramid with height OD = h and OD perpendicular to the plane of OAB as in the diagram above.

Consider a slice S of the pyramid of width δa as shown at EFGH in the diagram. The slice S is perpendicular to the plane of OAB at FG with FG || AB and BG = a. Note that GH || OD.

i) Show that the volume of S is $\left(\dfrac{r-a}{r}\right)b\left(\dfrac{ah}{r}\right)\delta a$ when δa is small.

(You may assume the slice is approximately a rectangular prism of base EFGH and height δa). [3]

ii) Hence show that the pyramid DOAB has a volume of $\dfrac{1}{6}hbr$. [2]

iii) Suppose now that $\angle AOB = \dfrac{2\pi}{n}$ and that n identical pyramids DOAB are arranged about O as the centre with common vertical axis OD to from a solid C. Show that the volume V_n of C is given by $V_n = \dfrac{1}{3}r^2 hn \sin\dfrac{\pi}{n}$. [2]

iv) Note that when n is large, C approximates a right circular cone. Hence, find $\lim\limits_{n\to\infty} V_n$ and verify a right circular cone of radius r and height h has a volume of $\dfrac{1}{3}\pi r^2 h$. [2]

b) On the hyperbola $xy = c^2$, three points P, Q and R are on the same branch, with parameters p, q and r respectively. The tangents at P and Q intersect at U. If O, U and R are collinear, find the relationship between p, q and r. [6]

<u>Question Eight:</u> (15 Marks) *Start a new sheet of paper.*

a)

i) Use the substitution $x = \dfrac{2}{3}\sin\theta$ to prove that $\displaystyle\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\, dx = \frac{\pi}{3}$. [3]

ii) Hence, or otherwise, find the area enclosed by the ellipse

$9x^2 + y^2 = 4$. [1]

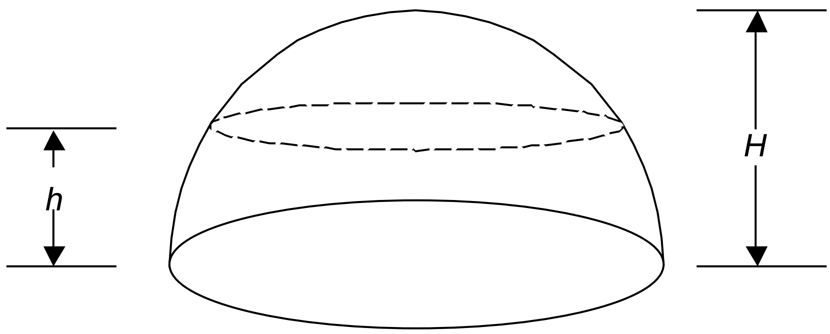
b)

i) Use an appropriate substitution to verify that $\displaystyle\int_0^{a} \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{4}$. [2]

ii) Deduce that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given

by πab. [2]

c) The diagram below shows a mound of height H. At height h above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\lambda^2$, where $\lambda = 1-\dfrac{h^2}{H^2}$, and x, y are appropriate coordinates in the plane of the cross-section.



Show that the volume of the mound is $\dfrac{8\pi abH}{15}$. [3]

d) The quadratic equation $x^2-(2\cos\theta)x+1=0$ has roots α and β.

i) Find expressions for α and β. [1]

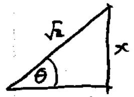
ii) Show that $\alpha^{10}+\beta^{10}=2\cos(10\theta)$. [3]

QUESTION ONE:

a) $x = \sqrt{2}\sin\theta$ so $dx = \sqrt{2}\cos\theta . d\theta$

$$\therefore \int \frac{x}{\sqrt{2-x^2}}\,dx$$
$$= \int \frac{\sqrt{2}\sin\theta . \sqrt{2}\cos\theta . d\theta}{\sqrt{2-(\sqrt{2}\sin\theta)^2}}$$
$$= \int \frac{2\sin\theta\cos\theta\,d\theta}{\sqrt{2(1-\sin^2\theta)}}$$
$$= \int \frac{\sqrt{2}\sin\theta\cos\theta\,d\theta}{\cos\theta}$$
$$= \int \sqrt{2}\sin\theta\,d\theta$$
$$= -\sqrt{2}\cos\theta + c$$
$$= -\sqrt{2} . \frac{\sqrt{2-x^2}}{\sqrt{2}} + c$$
$$= -\sqrt{2-x^2} + c$$

also



with $\sin\theta = \frac{x}{\sqrt{2}}$

$x^2 + y^2 = 2$

$y^2 = 2 - x^2$

$y = \sqrt{2-x^2}$

$\therefore \cos\theta = \frac{\sqrt{2-x^2}}{\sqrt{2}}$

① converting from x to θ, + resolving for $\cos\theta$

① answer

- Some got $dx = \sqrt{2}\cos\theta\,d\theta$, but then left it out of the substitution when changing to θ!!
- very common error was not expressing the indefinite integral back in terms of x; $-\sqrt{2}\cos\theta + c$ got 1 mark.

$-\sqrt{2}\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right) + c$ was also not sufficient for both marks (not simplest form).

b) $\sin(A+B) = \sin A\cos B + \sin B\cos A$ —①

$\sin(A-B) = \sin A\cos B - \sin B\cos A$ —②

$\sin(A+B) + \sin(A+B) = 2\sin A\cos B$ ①+②

$$\therefore \int \sin 5x \cos 3x\,dx$$
$$= \int \tfrac{1}{2}\left(\sin(5x+3x) + \sin(5x-3x)\right)dx$$
$$= \int \tfrac{1}{2}\sin 8x + \tfrac{1}{2}\sin 2x\,dx$$
$$= -\tfrac{1}{16}\cos 8x - \tfrac{1}{4}\cos 2x + c$$

① showing relationship

① correct use of formula

① Answer

- well done

c)
$$\int_0^1 \tan^{-1}x\,dx$$
$$= \int_0^1 \frac{d(x)}{dx} . \tan^{-1}x\,dx$$
$$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 x . \frac{1}{1+x^2}\,dx$$
$$= \left(\tfrac{\pi}{4} - 0\right) - \tfrac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}\left[\ln(1+x^2)\right]_0^1$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}(\ln 2 - \ln 1)$$
$$= \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

① correct integration by parts

① correct log integration

① correct algebra

- well done

d) i)
$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$
$$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x\,dx$$
$$= \left[\sin x\cos^{n-1}x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2}x . \sin x . \sin x\,dx$$
$$= (0-0) + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x\sin^2 x\,dx$$
$$= (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2}x\,dx$$
$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x\,dx$$
$$\therefore J_n = (n-1)J_{n-2} - (n-1)J_n$$
$$\therefore J_n + (n-1)J_n = (n-1)J_{n-2}$$
$$nJ_n = (n-1)J_{n-2}$$
$$\therefore J_n = \frac{(n-1)}{n}J_{n-2}$$

① correct method for splitting cos

① resolving to integral

① resolving to J_{n-2}

① correct algebra to solution

- many tried the approach $\int \frac{d(x)}{dx} . \cos^n x\,dx$ and got lost. These need to be known.

ii)
$$\therefore \int_0^{\frac{\pi}{2}} \cos^6 x\,dx = J_6$$
$$\therefore J_6 = \tfrac{5}{6} . J_4$$
$$= \tfrac{5}{6} . \tfrac{3}{4}J_2$$
$$= \tfrac{5}{6} . \tfrac{3}{4} . \tfrac{1}{2}J_0$$
$$= \tfrac{15}{48} . \int_0^{\frac{\pi}{2}} dx$$
$$= \tfrac{15}{48} . \left[x\right]_0^{\frac{\pi}{2}}$$
$$= \tfrac{15\pi}{96}$$

① correct use of formula

① evaluates J_0

① answer

- well done.

QUESTION TWO:

i) a) $(\bar{z})^2$

$= (2-i)^2$

$= 4 - 4i - 1$

$= 3 - 4i$

① answer

- some simple mistakes ma

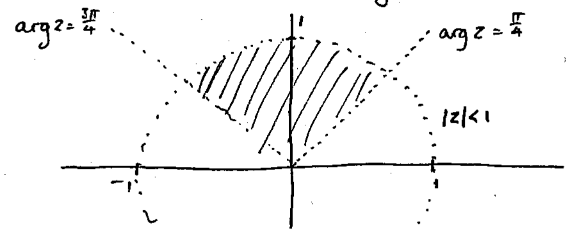
ii) $\left(\frac{z}{w}\right)$

$$= \frac{(2+i)}{(2-3i)} . \frac{(2+3i)}{(2+3i)}$$
$$= \frac{4+6i+2i-3}{4+9}$$
$$= \frac{1+8i}{13}$$

① answer

- generally good

) $0 < |z| < 1$ $\quad \frac{\pi}{4} < \arg z < \frac{3\pi}{4}$



① boundaries

① correct $|z|$

① correct arg limits.

- generally good.

c) $(a+ib) = \sqrt{7+6i\sqrt{2}}$ a, b real

$\therefore (a+ib)^2 = 7+6i\sqrt{2}$

$a^2-b^2+2abi = 7+6\sqrt{2}\,i$

equating real and imaginary parts.

$a^2-b^2=7$ — ① $2ab = 6\sqrt{2}$

$\therefore a = \frac{6\sqrt{2}}{2b} = \frac{3\sqrt{2}}{b}$ — ②

① set up a, b relationship

• mostly well done. Some students tried to use formulas for finding square roots of complex numbers (not very successfully)

substituting ② in ①:

$\left(\frac{3\sqrt{2}}{b}\right)^2 - b^2 = 7$

$\frac{18}{b^2} - b^2 = 7$

$18 - b^4 = 7b^2$

ie $b^4+7b^2-18=0$

$\therefore (b^2+9)(b^2-2)=0$

$b^2 = 2, -9$

① resolve for correct b values

reject $b^2=-9$ as b is real.

$\therefore b = \pm\sqrt{2}$ in ②:

$b=\sqrt{2}$: $a = \frac{3\sqrt{2}}{\sqrt{2}} = 3$ $b=-\sqrt{2}$: $a = \frac{3\sqrt{2}}{-\sqrt{2}} = -3$

① correct roots.

$\therefore$ roots are $3+\sqrt{2}\,i$, $-3-\sqrt{2}\,i$

d) i) for $z_1 \parallel z_2$, $\theta=\phi$, $\therefore z_2 = r_2 \operatorname{cis}\theta$

$\therefore \operatorname{cis}\theta = \frac{z_2}{r_2}$

• generally well done

$\therefore$ from $z_1 = r_1 \operatorname{cis}\theta = r_1 \cdot \frac{z_2}{r_2}$

$\therefore z_1 = k z_2$ where $k = \frac{r_1}{r_2}$

① deducing relationship.

ii) Side AB is either $\overrightarrow{AB}$ or $\overrightarrow{BA}$

$OA+AB=OB$ $OB+BA=OA$

$\therefore AB = OB-OA = z_2-z_1$ $\therefore BA = OA-OB = z_1-z_2 = -(z_2-z_1)$

① for both possibilities   • OK

iii) side CD is given by z_3-z_4 or (z_4-z_3)

as $CD \parallel AB$, $(z_4-z_3) = k(z_2-z_1)$

but opposite sides of a parallelogram are equal,

so $|z_4-z_3| = |z_2-z_1| \Rightarrow k=\pm1$

① side CD

$k=1$: $\therefore z_4-z_3 = z_2-z_1$

or $z_1-z_2-z_3+z_4=0$

① deriving k   • not well done.

$k=-1$: $\therefore z_4-z_3 = -(z_2-z_1)$

but from (ii), AB (or BA) can be either z_2-z_1 or $-(z_2-z_1)$

$\therefore z_4-z_3 = z_2-z_1$

$\therefore z_1-z_2-z_3+z_4=0$

① both cases for k.

iv) the argument uses only real numbers, not complex numbers, thus

$-1 = (a+ib)^2$, where a, b are real.

$= a^2-b^2+2abi$

$\therefore a^2-b^2=-1$ and $2ab=0$

$b=0 \Rightarrow a^2=-1$, which cannot happen as a is real

so $a=0$ and $-b^2=-1 \Rightarrow b=\pm1$

$\therefore$ the roots are $-i$ and i

$i^2=-1$ and $(-i)^2 = (-1)^2 i^2 = -1$

$\sqrt{-1}\times\sqrt{-1} = \sqrt{1\times-1}\times\sqrt{1\times-1} = 1i\times1i \ne \sqrt{(-1)\times(-1)}$.

① identifies use of real nos to try and solve a complex no problem

① demonstrates correct procedure (in some way).

• Some good answers, most realized something was wrong, but couldn't articulate what it was

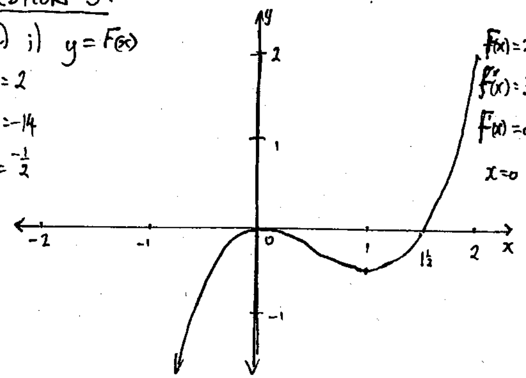
QUESTION 3:

a) i) $y=F(x)$

$F(2)=2$

$F(-2)=-14$

$F(1)=-\frac{1}{2}$



$F(x)=x^3-\frac{3}{2}x^2$

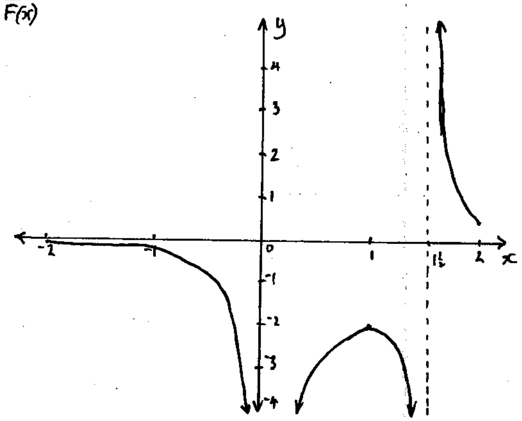
$F'(x)=3x^2-3x$

$F(x)=0$ gives $x=0, x=1$

① axis intercepts

① turning pt. @ $(1,-\frac{1}{2})$

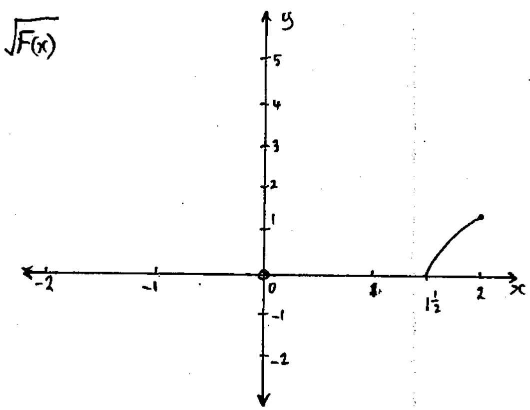
ii) $y = \frac{1}{F(x)}$



① asymtotes

① correct shape in areas.

iii) $y = \sqrt{F(x)}$

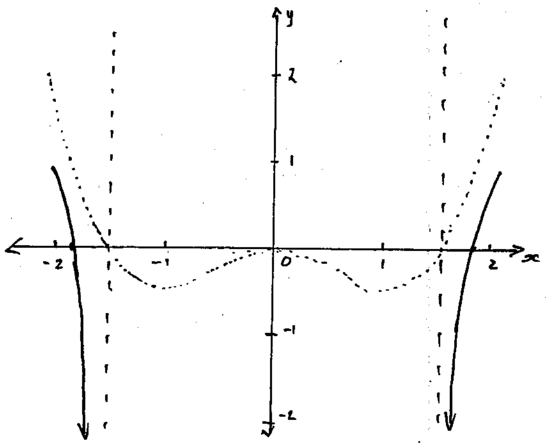


① correct intercept

① correct shape in areas

• no penalty for (0,0) not plotted

iv)



① asymptotes

① correct shape in area.

b) i) $a = 5$, $b = 3$ and for hyperbola: $b^2 = a^2(e^2 - 1)$

$\therefore\ 9 = 25(e^2 - 1)$

$e^2 - 1 = \frac{9}{25}$

$e^2 = \frac{34}{25}$

$\therefore e = \frac{\sqrt{34}}{5}$

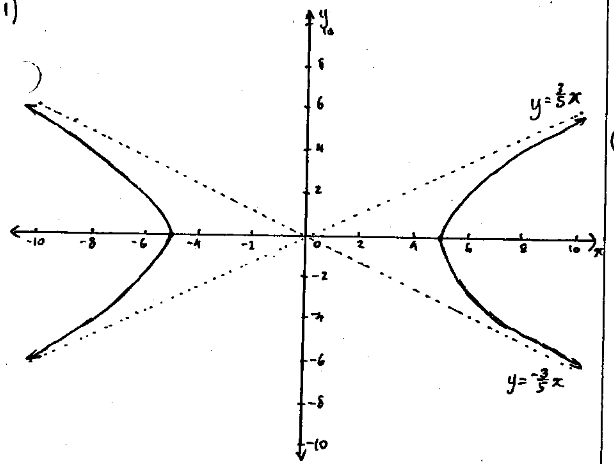
① correct value of e

ii) Foci are $S(ae, 0)$ and $S'(-ae, 0)$

$\therefore\ (\sqrt{34}, 0)$ and $(-\sqrt{34}, 0)$

① correct foci.

iii)



① asymptotes

① correct shape + intercepts.

• could also get this mark if shape is reasonable & scale indicated on y-ax

iv) $x = 6$: $\frac{36}{25} - \frac{y^2}{9} = 1$

$\frac{36}{25} - 1 = \frac{y^2}{9}$

$y^2 = \frac{9 \times 11}{25}$

$\therefore\ y = \pm\frac{\sqrt{99}}{5}$

$\therefore$ A is $\left(6, \frac{\sqrt{99}}{5}\right)$ and B is $\left(6, -\frac{\sqrt{99}}{5}\right)$

① correct A and B

v) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

$\therefore\ \frac{2x}{25} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$

so $\frac{dy}{dx} = \frac{-2x}{25} \cdot \frac{-9}{2y}$

$= \frac{9x}{25y}$

at $\left(6, \frac{\sqrt{99}}{5}\right)$, $\frac{dy}{dx} = \frac{9 \cdot 6 \cdot 5}{25 \cdot \sqrt{99}}$

$= \frac{54}{5\sqrt{99}}$

① correct differentiation for $\frac{dy}{dx}$

① correct subst to eqn.

• ignored small arithmetic errors.

v) (cont) $\therefore\ y - \frac{\sqrt{99}}{5} = \frac{54}{5\sqrt{99}}(x-6)$

$5\sqrt{99}\,y - 99 = 54x - 324$

$\therefore\ 0 = 54x - 5\sqrt{99}\,y - 225$

QUESTION 4:

a) i) at P: $3 - x^2 = x^2 - x$

$\therefore\ 0 = 2x^2 - x - 3$

$= 2x^2 + 2x - 3x - 3$

$= 2x(x+1) - 3(x+1)$

$= (x+1)(2x-3)$

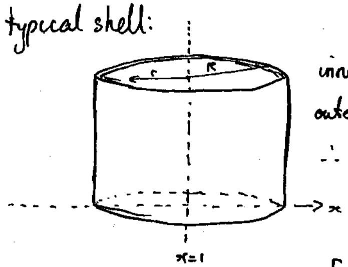
$\therefore\ x = -1,\ 3/2$

$\therefore$ x co-ord of P is -1 (as P is in 2nd quadrant)

Marking: ① correct value for x.

---

ii) typical shell:



inner radius: $r = 1 - x$

outer Radius: $R = 1 - (x + \delta x)$

$\therefore$ Area of annulus:

$\delta A = \pi R^2 - \pi r^2$

$= \pi\left(1-(x+\delta x)\right)^2 - \pi(1-x)^2$

$= \pi\left[1 - 2(x+\delta x) + (x+\delta x)^2 - (1 - 2x + x^2)\right]$

$= \pi\left[1 - 2x - 2\delta x + x^2 + 2x\delta x + \delta x^2 - 1 + 2x - x^2\right]$

$= \pi(2x\delta x - 2\delta x + \delta x^2)$

$= 2\pi(x-1)\delta x$ (ignoring δx^2 as too small).

Marking: ① area of annulus δA

$\therefore$ a small volume of shell is given by

$\delta V = \delta A . h$ where $h = (3 - x^2) - (x^2 - x)$

$= 3 + x - 2x^2$

Marking: ① correct h leading to δV

$\therefore\ \delta V = 2\pi(x-1)(3 + x - 2x^2)\delta x$

$= 2\pi(3x + x^2 - 2x^3 - 3 - x + 2x^2)\delta x$

$= 2\pi(-3 + 2x + 3x^2 - 2x^3)\delta x$

$\therefore$ Volume of the solid is given by

$V = \sum \delta V$

$= \lim_{\delta x \to 0} \sum_{x=-1}^{1} 2\pi(-3 + 2x + 3x^2 - 2x^3)\delta x$

$= 2\pi \int_{-1}^{1} -3 + 2x + 3x^2 - 2x^3\, dx$

Marking: ① correct summing leading to integral.

Comments: alternatively:
1 mark: volume of typical shell
1 mark: correct limits
1 mark: integration

iii) $\therefore\ V = 2\pi\left|\left[-3x + x^2 + x^3 - \tfrac{1}{2}x^4\right]_{-1}^{1}\right|$

$= 2\pi\left|\left[(-3+1+1-\tfrac{1}{2}) - (3+1-1-\tfrac{1}{2})\right]\right|$

$= 2\pi\left| -1\tfrac{1}{2} - 2\tfrac{1}{2}\right|$

$= 8\pi$

Marking: ① correct integration ① correct answer

Comments: · Full marks only for cor[rect] solution.

---

b) $\therefore\ \frac{x}{(x-1)^2(x-2)} = \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$

$\therefore\ x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$ —①

in ①: $x = 1$ gives: $1 = B(1-2)$ ie $B = -1$

$x = 2$ gives: $2 = C(2-1)^2$ ie $C = 2$

Marking: ① subst to find B, C (or any other method)

also, from ①: $x = A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1)$

equating co-efficients of x^2:

$0 = A + C$

$\therefore\ A = -2$

Marking: ① equating co-effs to find A.

$\therefore\ \frac{x}{(x-1)^2(x-2)} = \frac{-2}{(x-1)} - \frac{1}{(x+1)^2} + \frac{2}{x-2}$

$\therefore\ \int_0^{1/2} \frac{x}{(x-1)^2(x-2)}\, dx$

$= \int_0^{1/2} \frac{2}{(x-2)} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\, dx$

$= \int_0^{1/2} \frac{-2}{2-x} + \frac{2}{1-x} - \frac{1}{(x-1)^2}\, dx$

Marking: ① correct rearrangement to get to integration

$= \left[-2\ln(2-x).(-1) + 2\ln(1-x).(-1) + \frac{1}{x-1}\right]_0^{1/2}$

$= \left[2\ln(2-x) - 2\ln(1-x) + \frac{1}{x-1}\right]_0^{1/2}$

$= 2\ln\frac{3}{2} + 2\ln 2 - 2 - (2\ln 2 - 2\ln 1 - 1)$

$= 2\ln\frac{3}{2} + 2\ln 2 - 2 - 2\ln 2 + 0 + 1$

$= 2\ln\left(\frac{3}{2}\right) - 1.$

Marking: ① correct subst to show answer.

---

c) $\cos 2x = c = \sin 3x$ $\quad$ c constant

$\therefore\ \sin 3x = c$

or $\cos\left(\frac{\pi}{2} - 3x\right) = c$

$\frac{\pi}{2} - 3x = \cos^{-1}(c) + 2\pi n \quad n = 0, \pm 1, \pm 2 \ldots$

c) (cont) $\therefore\ -3x = -\frac{\pi}{2} + 2\pi n + \cos^{-1}(c)$

or $3x = \frac{\pi}{2} - 2\pi n - \cos^{-1}(c)$

but $\cos 2x = c$ also,

so $2x = \cos^{-1}(c)$

$\therefore\ 3x = \frac{\pi}{2} - 2\pi n - 2x$

$5x = \left(\frac{1-4n}{2}\right)\pi$

$\therefore\ x = \left(\frac{1-4n}{10}\right)\pi$

for $0 \le x \le \frac{\pi}{2}$, we get (using $n=0, n=-1$)

$x = \frac{\pi}{10}, \frac{\pi}{2}$

Marking: ① correct setup of problem (any method); ① correct solutions in range.

---

d) let $\tan^{-1}3x = \theta$ and $\tan^{-1}2x = \phi$

$\therefore\ \tan\theta = 3x \qquad \tan\phi = 2x$

for $\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}(3x) - \tan^{-1}(2x)$

$= \theta - \phi$

taking tan of both sides:

$\tan\left(\tan^{-1}\frac{1}{5}\right) = \tan(\theta - \phi)$

$\therefore\ \frac{1}{5} = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

$= \frac{3x - 2x}{1 + 3x.2x}$

$1 + 6x^2 = 5x$

or $0 = 6x^2 - 5x + 1$

$= 6x^2 - 3x - 2x + 1$

$= 3x(2x-1) - 1(2x-1)$

$= (2x-1)(3x-1)$

$\therefore\ x = \frac{1}{2}, \frac{1}{3}$

Marking: ① correct use of tan; ① forms quadratic; ① correct answers

---

QUESTION 5:

a) $\alpha + \beta + \gamma = -4$

$\alpha\beta + \alpha\gamma + \beta\gamma = 2$

$\alpha\beta\gamma = 3$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (-4)^2 - 2(2)$

$= 12$

Marking: ① answer

Comments: • some had an incorrect squares expansion!

---

ii) for roots $x = 1 - \alpha \implies \alpha = (1 - x)$

$\therefore$ $(1-x)$ in eqn gives:

$(1-x)^3 + 4(1-x)^2 + 2(1-x) - 3 = 0$

$1 - 3x + 3x^2 - x^3 + 4 - 8x + 4x^2 + 2 - 2x - 3 = 0$

$\therefore\ 4 - 13x + 7x^2 - x^3 = 0$

or $x^3 - 7x^2 + 13x - 4 = 0$

Marking: ① correct setup with roots; ① correct eqn.

Comments: • many simple algebraic errors

---

iii) for roots $x = \frac{1}{\alpha} \implies \alpha = \frac{1}{x}$

$\therefore\ \left(\frac{1}{x}\right)^3 + 4\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) - 3 = 0$

$\times x^3$: $1 + 4x + 2x^2 - 3x^3 = 0$

or $3x^3 - 2x^2 - 4x - 1 = 0$

Marking: ① setup with roots; ① correct eqn.

---

b) let α be the root of multiplicity 3,

then $P(\alpha) = P'(\alpha) = P''(\alpha) = 0$.

$\therefore\ P'(x) = 32x^3 - 75x^2 + 54x - 11$

$P''(x) = 96x^2 - 150x + 54$

if $P''(\alpha) = 0$, α is the soln to $0 = 96x^2 - 150x + 54$

or $0 = 48x^2 - 75x + 27$

$\therefore\ x = \frac{75 \pm \sqrt{2625 - 4.48.27}}{96}$

$= \frac{75 \pm \sqrt{441}}{96}$

$= \frac{75 \pm 21}{96}$

$= 1, \frac{9}{16}$

now, $P'(1) = 32 - 75 + 54 - 11$

$= 0$

and $P(1) = 8 - 25 + 27 - 11 + 1$

$= 0$

$\therefore\ (x-1)^3$ is a factor of $P(x)$

so $\alpha = 1$ is the tripple root.

Also, $\alpha^3.\beta = \frac{1}{8}$

$\therefore\ \beta = \frac{1}{8}$ is the other root.

Marking: ① set up problem with $P''(\alpha) = 0$; ① correct possibilities for tripple root; ① correct tripple root with reasons; ① other root

Comments: • several used $P'''(\alpha) = 0$ • many did not understand the implications for a root with multiplicity! • need to explicitly state the other root. $(8x-1)$ as a factor implies a root of $x = \frac{1}{8}$.

---

c) let α, β, γ and δ be the roots of the equation

i) $\therefore\ x = \alpha + 1$ is a root of the reqd eqn

so $\alpha = x - 1$

$\therefore\ (x-1)^4 + 4(x-1)^3 + 5(x-1)^2 + 2(x-1) - 20 = 0$

Marking: ① correct subst for root.

i) (cont):

$(x^4-4x^3+6x^2-4x+1)+(4x^3-12x^2+12x-4)+(5x^2-10x+5)+2x-2-20=0$

$\therefore x^4-x^2-20=0$ as reqd.

① correct alg. to soln.

• many errors expanding this

ii) now $x^4-x^2-20=0$

$(x^2-5)(x^2+4)=0$

$\therefore x^2=5,-4$

$\therefore x=\pm\sqrt{5}, \pm 2i$

$\therefore$ the roots of $x^4+4x^3+5x^2+2x-20=0$ are given by $\alpha = x-1$

$\therefore$ roots are $-1+\sqrt{5}, -1-\sqrt{5}, -1+2i, -1-2i$

① correct roots for $x^4-x^2-20=0$

① correct roots for orig. eqn.

• many had trouble linking the roots back to the original with $\alpha = x-1$

QUESTION 6:

a) i) $x_P = OS$ $\quad y_P = OT$

$= 5\cos\theta \quad = 3\sin\theta$

①

① each answer

• link back to the definition given in the diagram. This is the starting point, and many missed it.

ii) $\therefore \frac{x}{5}=\cos\theta$ and $\frac{y}{3}=\sin\theta$

$\frac{x^2}{25}=\cos^2\theta \quad \frac{y^2}{9}=\sin^2\theta$

$\therefore \frac{x^2}{25}+\frac{y^2}{9}=\cos^2\theta+\sin^2\theta$

so $\frac{x^2}{25}+\frac{y^2}{9}=1$ is the cartesian eqn.

① correct eqn.

• "eliminate θ" $\Rightarrow$ show how this happens, don't just write the equation down!

iii) normal to an ellipse is given by

$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ where $a=5, b=3$, $\theta=\frac{\pi}{3}$

$\therefore \frac{5x}{\cos\frac{\pi}{3}}-\frac{3y}{\sin\frac{\pi}{3}}=25-9$

$10x-\frac{6y}{\sqrt{3}}=16$

$\therefore 10\sqrt{3}x-6y-16\sqrt{3}=0$ is eqn.

① correct subst in formula

① correct eqn (any form)

• many found tangent instead of normal.

• put your answer into one of the standard simplest forms - many left their answer unfinished

b) i) $S'(x)=\frac{d}{dx}\left(\frac{1}{2}(e^x-e^{-x})\right)$

$=\frac{1}{2}(e^x+e^{-x})$

$=C(x)$

① set out clearly.

ii) $e^x>0$ for all x

$e^{-x}>0$ for all x

$\therefore e^x+e^{-x}>0$ for all x

$\therefore C(x)>0$ for all x

ie $S'(x)>0$ for all x

$\Rightarrow S(x)$ is monotonically increasing

① correct reasoning.

• asked to show $\Rightarrow$ give reasons why $S'(x)>0$. Just stating it earns no marks.

iii) $[C(x)]^2=\left[\frac{1}{2}(e^x+e^{-x})\right]^2$

$=\frac{1}{4}(e^{2x}+2e^xe^{-x}+e^{-2x})$

$=\frac{1}{4}(e^{2x}+e^{-2x}+2)$

$1+[S(x)]^2=1+\left[\frac{1}{2}(e^x-e^{-x})\right]^2$

$=1+\frac{1}{4}(e^{2x}-2e^xe^{-x}+e^{-2x})$

$=\frac{1}{4}(4+e^{2x}-2+e^{-2x})$

$=\frac{1}{4}(e^{2x}+e^{-2x}-2)$

$=[C(x)]^2$ from above

① expression for $[C(x)]^2$ correct

① reduction of $1+[S(x)]^2$ correct (or equivalent)

iv) as $S(x)$ is monotonically increasing, each x must produce a unique y value

$\Rightarrow S(x)$ has a 1-1 correspondence

$\therefore S^{-1}(x)$ exists for all values of x.

① appropriate explanation

• many attempted explanati[ons] revealed a lack of underst[anding] of what inverse means.

v) $y=S^{-1}(x)$

$\therefore S(y)=x$

$\therefore \frac{dx}{dy}=S'(y)$

$=C(y)$

$=\sqrt{1+[S(y)]^2}$

$=\sqrt{1+x^2}$

$\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}$

① inverse rules to give $\frac{dx}{dy}$ correctly

① correct subst to formula.

• very few picked up the lin[k] to $S(x)$ and $C(x)$ in the previ[ous] parts, so many futile a[ttempts] at a simple problem. Look [for] linked parts like this one - make the solution simple

v') $\therefore y=\int\frac{dx}{\sqrt{1+x^2}}$ $\quad$ let $x=\tan\theta$

$\therefore dx=\sec^2\theta\, d\theta$

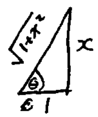
$=\int\frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$

$=\int\frac{\sec^2\theta\, d\theta}{\sec\theta}$

$=\int\sec\theta\, d\theta$

$=\int\frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\,d\theta$

$=\ln(\sec\theta+\tan\theta)+c$



$\therefore y=\ln(x+\sqrt{1+x^2})+c$

① reduction to $\int\sec\theta\, d\theta$

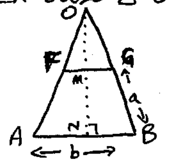
① correct $\int$ of $\sec\theta$

① correct subst to give y in terms of x.

• The question is to show t[he] relationship $\Rightarrow$ not us[e] the standard integral table. T[he] integration is the questio[n]; its not part of something bigger.

QUESTION 7:

a) i) In base Δ OAB:



$GB=a \quad OB=r$

$NB=\frac{b}{2} \quad OG=r-a$

① correct setup of variables

a) (cont) $\therefore \frac{MG}{OG} = \frac{NB}{OB}$

or $MG = \frac{NB \cdot OG}{OB}$

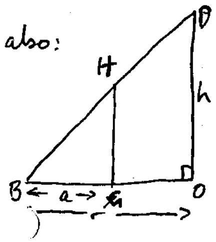
$= \frac{b}{2} \cdot \frac{(r-a)}{r}$

$\therefore FG = 2MG$

$= \frac{b(r-a)}{r}$

① correct value for FG

also:



$OD = h, \quad OB = r$

$GB = a$

$\therefore \frac{GH}{GB} = \frac{OD}{OB}$

$\therefore GH = \frac{OD \cdot GB}{OB}$

$= \frac{a \cdot h}{r}$

① correct expression for GH.

$\therefore V_s = FG \cdot GH \cdot \delta a$

$= \left(\frac{r-a}{r}\right) b \cdot \left(\frac{ah}{r}\right) \delta a$

ii) $\therefore V = \int_0^r \left(\frac{r-a}{r}\right) b \cdot \left(\frac{ah}{r}\right) da$

$= \frac{bh}{r^2} \int_0^r a(r-a)\, da$

$= \frac{bh}{r^2} \int_0^r ar - a^2\, da$

$= \frac{bh}{r^2} \left[\frac{1}{2}a^2 r - \frac{1}{3}a^3\right]_0^r$

$= \frac{bh}{r^2} \left[\left(\frac{1}{2}r^3 - \frac{1}{3}r^3\right) - 0\right]$

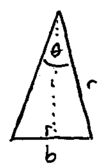
$= \frac{bh}{r^2} \cdot \left(\frac{1}{2} - \frac{1}{3}\right) r^3$

$= \frac{1}{6} bhr$ as reqd.

① reduction to correct $\int$

① correct subst to expression

iii) given $\angle AOB = \frac{2\pi}{n}$



ie $\theta = \frac{2\pi}{n}$

$\therefore \frac{\theta}{2} = \frac{\pi}{n}$

$\therefore \sin\frac{\theta}{2} = \frac{b}{2} \cdot \frac{1}{r}$

or $b = 2r \sin\frac{\theta}{2}$

$= 2r \sin\frac{\pi}{n}$

① correct derivation for b.

$\therefore V = \frac{1}{6} bhr$ from (ii) above

so $V = \frac{1}{6} hr \cdot 2r \sin\frac{\pi}{n}$

$= \frac{1}{3} hr^2 \sin\frac{\pi}{n}$

ie $V_n = \frac{1}{3} hr^2 n \sin\frac{\pi}{n}$

① correct expression for V.

p13

iv) (cont) $\therefore \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{1}{3} nhr^2 \sin\frac{\pi}{n}$

$= \lim_{n\to\infty} \frac{1}{3} hr^2 \pi \cdot \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}$

let $x = \frac{\pi}{n}$; as $n \to \infty$ $\frac{\pi}{n} \to 0$

$\therefore \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{1}{3} hr^2 \pi \cdot \lim_{x\to 0} \frac{\sin x}{x}$

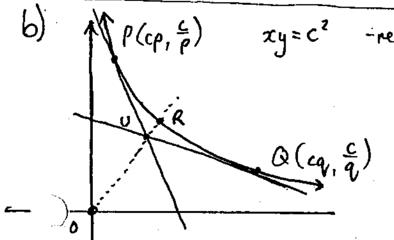
$= \frac{1}{3}\pi r^2 h$

① limit expression

① correct use of limits to evaluate expression

• many incorrect uses of the limit.

b)



$xy = c^2$ - rectangular hyperbola

① diagram showing relationships

• many students didn't draw correct diagrams!

tangent at P: $x + p^2 y = 2cp$ — ①

Q: $x + q^2 y = 2cq$ — ②

$(p^2 - q^2) y = 2c(p-q)$: ①-②

$\therefore y_U = \frac{2c}{p+q}$

① finding y co-ord of U

• not many students completed the correct relationship.

$\therefore x_U = 2cp - p^2 \frac{2c}{p+q}$

$= \frac{2cpq}{p+q}$

① finding x co-ord of U

• Some became lost after getting correct points of intersection, others made it much more complicated than it was.

Now $m_{OU} = \frac{2c}{p+q} \cdot \frac{p+q}{2cpq}$

$= \frac{1}{pq}$

① gradient OU

$\therefore$ eqn of OUR is $y = \frac{x}{pq}$ — ③

$xy = c^2$ — ④

subst ③ in ④: $\frac{x^2}{pq} = c^2$

$\therefore x^2 = pqc^2$

$\therefore x_R = c\sqrt{pq}$

① finding x_R (or y_R)

but R is $(cr, \frac{c}{r})$

so $cr = c\sqrt{pq}$

$r = \sqrt{pq}$

or $r^2 = pq$

① correct relationship (either form)

QUESTION 8:

a) i) $\int_0^{2/3}\sqrt{4-9x^2}\,dx$

$x=\frac{2}{3}\sin\theta$

$\therefore dx=\frac{2}{3}\cos\theta\,d\theta$

when $x=\frac{2}{3}$, $\theta=\frac{\pi}{2}$

$x=0$, $\theta=0$

① correct subst to θ, including limits

$\therefore \; = \int_0^{\frac{\pi}{2}}\sqrt{4-4\sin^2\theta}\cdot\frac{2}{3}\cos\theta\,d\theta$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$

① correct reduction to $\int\cos^2$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos 2\theta+1)\,d\theta$

① correct subst to soln.

$= \frac{2}{3}\left[\frac{1}{2}\sin 2\theta+\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{3}\left[\left(\frac{1}{2}\cdot 0+\frac{\pi}{2}\right)-0\right]$

$= \frac{\pi}{3}$

ii) for $9x^2+y^2=4$

$y^2=4-9x^2$

$\therefore y=\sqrt{4-9x^2}$

$\therefore$ pt i) gives the area in the first quadrant so, from symmetry, this is $\frac{1}{4}$ the reqd. area.

$\therefore A=4\cdot\frac{\pi}{3}$

$=\frac{4\pi}{3}$ sq units.

① answer.

b) i) $x=a\sin\theta$ $\quad\therefore dx=a\cos\theta\,d\theta$

when $x=a$, $\theta=\frac{\pi}{2}$

$+ \; x=0$, $\theta=0$

$\therefore \int_0^a\sqrt{a^2-x^2}\,dx$

$= \int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

① correct subst inc limits

$= a^2\int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$

$= a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

① correct integration to solution

$= a^2\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos 2\theta-1)\,d\theta$

$= a^2\left[\frac{1}{4}\sin 2\theta-\theta/2\right]_0^{\pi/2}$

$= a^2\left[\left(0+\frac{\pi}{4}\right)-0\right]$

$= \frac{a^2\pi}{4}$

ii) from $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$b^2x^2+a^2y^2=a^2b^2$

$a^2y^2=a^2b^2-b^2x^2$

$\therefore y^2=b^2-\frac{b^2}{a^2}x^2$

$\therefore y=\sqrt{b^2-\frac{b^2}{a^2}x^2}$

$=\sqrt{\frac{b^2a^2-b^2x^2}{a^2}}$

$=\frac{b}{a}\sqrt{a^2-x^2}$

① reducing equation to stnd. form

$\therefore$ area of 1st quadrant is

$A_1=\int_0^a\frac{b}{a}\sqrt{a^2-x^2}\,dx$

① correct reasoning to soln.

$=\frac{b}{a}\cdot\frac{\pi a^2}{4}$ from (i)

$=\frac{\pi ab}{4}$

$\therefore$ total area (from symmetry)

$A=4\cdot\frac{\pi ab}{4}$

$=\pi ab$

c) $\Delta V=A\,\Delta h$ where A is the area of the ellipse at height h.

$\therefore$ from (b) above:

$A=\pi ab\lambda^2$

(as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda^2$ becomes $\frac{x^2}{a^2\lambda^2}+\frac{y^2}{b^2\lambda^2}=1$)

① correct deduction of A

$\therefore \Delta V=\pi ab\lambda^2\Delta h$

$\therefore V=\int_0^H\pi ab\left(1-\frac{h^2}{H^2}\right)^2dh$

① correct expression for V in terms of h's.

• wrong λ, but correct method, gained 1 mark

$=\pi ab\int_0^H 1-\frac{2h^2}{H^2}+\frac{h^4}{H^4}\,dh$

$=\pi ab\left[h-\frac{2}{3}\frac{h^3}{H^2}+\frac{h^5}{5H^4}\right]_0^H$

① correct $\int$ leading to soln

$=\pi ab\left[\left(H-\frac{2}{3}\frac{H^3}{H^2}+\frac{H^5}{5H^4}\right)-0\right]$

$=\pi ab\left[\frac{15H-10H+3H}{15}\right]$

$=\frac{8\pi abH}{15}$ as reqd.

d) $x^2-(2\cos\theta)x+1=0$

i) $x^2-(2\cos\theta)x+\cos^2\theta=-1+\cos^2\theta$

$\therefore (x-\cos\theta)^2=-\sin^2\theta$

$\therefore x-\cos\theta=\pm i\sin\theta$

$\therefore x=\cos\theta\pm i\sin\theta$

$\therefore \alpha=\cos\theta+i\sin\theta \quad \beta=\cos\theta-i\sin\theta$

ii) $\alpha=\text{cis}\,\theta$

$\therefore \alpha^{10}=(\text{cis}\,\theta)^{10}$

$=\text{cis}\,10\theta$ by deMoivres Theorem

similarly $\beta=\overline{\text{cis}\,\theta}$

so $\beta^{10}=(\overline{\text{cis}\,\theta})^{10}$

$=\overline{\text{cis}\,10\theta}$

$\therefore \alpha^{10}+\beta^{10}=\text{cis}\,10\theta+\overline{\text{cis}\,10\theta}$

$=\cos 10\theta+i\sin 10\theta+\cos 10\theta-i\sin 10\theta$

$=2\cos 10\theta$ as reqd.

MARKING
both (1) answers
(1) correct use of deMoivres Theorem
(1) correct use of $\overline{\text{cis}\,\theta}$
(1) correct algebra to soln.